Constructing Arbitrary Positive Hausdorff Dimensions via Smale Horseshoe

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Abstract: We construct, for any $d_{target} > 0$, a subset of a Euclidean space with Hausdorff dimension d_{target}. The fractional part is realized by a linear, symmetric two-strip Smale horseshoe on $[0, 1]^2$ with expansion $\lambda > 2$ (horizontal contraction $1/\lambda$), $C^{1+\alpha}$-smoothed off the invariant set; in this model the invariant set has dimension $D(\lambda) = 2\ln2/\ln\lambda$, a continuous, strictly decreasing map with range (0, 2). The integer part follows from $\dim_H (A \times [0, 1]^n) = \dim_H (A) + n$. We briefly recall the needed tools and give explicit examples.

1. Introduction

In classical Euclidean geometry, the dimension of a set is a non-negative integer, adequately describing smooth subsets, polyhedra, and other regular objects. However, sets generated by iterative processes in dynamical systems, such as invariant sets under specific maps, often exhibit intricate self-similar or self-affine structures that defy integer-dimensional classification. These sets, characterized by scaling behaviors across multiple scales, necessitate a generalized notion of dimension that can take non-integer values. While some dynamical systems may produce integer-dimensional sets, such as periodic orbits, our focus is on complex sets with fractal properties. The development of fractal geometry and geometric measure theory provides a rigorous framework to assign real-valued dimensions to such sets, enabling precise quantification of their complexity in metric spaces.

The foundation for generalized dimensions was established by Hausdorff in his seminal 1918 paper[3], introducing Hausdorff measure and dimension. By defining a measure based on coverings with sets of arbitrary diameter, Hausdorff formalized fractional dimensions for any set in a metric space. This framework, refined by Abram Besicovitch and others, is sometimes called the Hausdorff-Besicovitch dimension[2]. Initially theoretical curiosities, non-integer dimensional sets gained prominence with dynamical systems theory, which provided systematic mechanisms for their generation.

In 1967, Smale's survey outlined the horseshoe mechanism: a surface diffeomor-phism that stretches, folds, and re-inserts a rectangle to create a totally disconnected hyperbolic invariant set (a "horseshoe")[10]. This provided a clean bridge between chaotic dynamics and fractal geometry. In parallel, dissipative models with so-called strange attractors—notably the H ́enon map and the Lorenz flow—motivated quantitative notions of complexity via fractal dimensions; here rigorous results primarily concern existence (e.g., [1] for H ́enon;for Lorenz), while most reported dimension values are numerical or refer to non-Hausdorff notions. The Kaplan–Yorke formula [5] is a widely used conjectural estimate from Lyapunov exponents. By contrast, for uniformly hyperbolic sets such as horseshoes, thermodynamic formalism yields rigorous formulas: on surfaces, Ma˜n´e proved the additivity

$$\dim_H (\Lambda) = d^s + d^u ,$$

where $d^{s/u}$ are given by pressure equations [6,]. In our linear two-strip model this reduces to $d^s = d^u = \ln2/\ln\lambda$.

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To construct sets with higher dimensions, the behavior of Hausdorff dimension under Cartesian products is essential. Classical results, going back to Marstrand, provide bounds rather than a general identity: for Borel (or analytic) sets $A \subset R^m$ and $B \subset R^n$,

$$\dim_H (A\times B) \geq \dim_H (A)+\dim_H (B) \text{ and } \dim_H (A\times B) \leq \dim_H (A)+\dim_B (B),$$

see Falconer [2, Ch. 7, Product formulae 7.2–7.3]; cf. [7]. Equality need not hold in general. However, since $\dim_H ([0, 1]^n) = \dim_B ([0, 1]^n) = n$, Falconer [2, Cor. 7.4] yields the identity $\dim_H (A \times [0, 1]^n) = \dim_H (A) + n$, which is the only product case we will use (formalized below as Theorem 2.8 and Corollary 2.9). In parallel, Moran and Hutchinson formalized the dimension of self-similar sets via the Moran– Hutchinson equation under the open set condition [2,8]. These tools enable precise dimension computations in dynamical and geometric contexts.

This paper synthesizes these concepts to constructively prove that any positive real number $d_{target} > 0$ can be the Hausdorff dimension of a set in a metric space. The main theorem (Theorem 3.2) decomposes d_{target} into an integer part $n = \lfloor d_{target} \rfloor$ and a fractional part $d_{frac} \in [0, 1)$. The fractional part is realized by tuning the expansion parameter of a Smale horseshoe map, whose invariant set's dimension is a continuous function on (0, 2) (Proposition 3.1). The integer part is contributed by a Euclidean hypercube $[0, 1]^n$. The final set, formed as their Cartesian product, has its dimension verified by the product rule (Theorem 2.8). Explicit examples for dimensions like $\sqrt{2}$, π, and a near-boundary case are provided, alongside a discussion of alternative dynamical generators, emphasizing the horseshoe's simplicity and explicit parameter dependence.

This paper is organized as follows. Section 2 reviews the essential concepts of Hausdorff measure, dimension, self-similar sets, and the Smale horseshoe map. Section 3 presents the main theorem, proving the tunability of the horseshoe's invariant set dimension and constructing sets with arbitrary positive dimensions. Section 4 provides explicit constructions for specific dimensions, including typical non-integer and near-boundary cases. Section 5 discusses alternative fractal generators and potential extensions, highlighting the method's modularity and future research directions.

2. Preliminaries

This section reviews the essential concepts from geometric measure theory and dynamical systems that underpin the main construction of this paper. We introduce the definitions and key results for Hausdorff measure and dimension, the dimension theory of self-similar sets generated by iterative maps, and the Smale horseshoe map with its hyperbolic invariant set. These tools provide the mathematical framework to construct sets with arbitrary positive real Hausdorff dimensions. The primary reference for standard definitions and results is the comprehensive textbook by Falconer [2]{Moran, 1946 #8}.

2.1. Hausdorff Measure and Dimension

Hausdorff measure and dimension give a rigorous way to assign real-valued dimen- sions to sets in metric spaces, including invariant sets in dynamical systems.

Definition 2.1 (Hausdorff outer measure). Let $S \subset R^k$, $d \geq 0$, and $\delta > 0$. The δ-approximate d-dimensional Hausdorff outer measure is

$$\mathcal{H}^d_\delta(S) = \inf\Big\{ \sum_{i=1}^{\infty} \big(\operatorname{diam} U_i\big)^d : \ S \subseteq \bigcup_{i=1}^{\infty} U_i, \ \operatorname{diam} U_i \leq \delta \Big\}.$$

The d-dimensional Hausdorff measure is

$$\mathcal{H}^d(S) = \sup_{\delta>0} \mathcal{H}^d_\delta(S) = \lim_{\delta\downarrow 0} \mathcal{H}^d_\delta(S).$$

We use the unnormalized version (no dimensional constants), which does not affect the value of the Hausdorff dimension.

The definition above is standard in fractal geometry. Alternative formulations, such as the spherical Hausdorff measure, include a normalizing constant (e.g., $c_d = \pi^{d/2}/\Gamma(d/2+1)$) to align with

Lebesgue measure in integer dimensions. Since this paper focuses on Hausdorff dimension, where constant factors do not affect the critical exponent, the simpler form suffices.

Definition 2.2 (Hausdorff dimension). The Hausdorff dimension of S is the thresh- old at which the Hausdorff measure drops from ∞ to 0:

$$\dim_H(S) = \inf\{d \geq 0 : H^d(S) = 0\} = \sup\{d \geq 0 : H^d(S) = \infty\}.$$

These definitions allow non-integer dimensions for sets such as Cantor-like invari- ant sets generated by hyperbolic maps. For example, countable sets have $\dim_H = 0$, whereas any n-dimensional set with positive Lebesgue measure in Rn has $\dim_H = n$.

2.2. Dimension of Self-Similar and Product Sets

To compute the Hausdorff dimension of fractal sets generated by iterative maps, such as the invariant set of the Smale horseshoe, we rely on the theory of self-similar sets. This framework, pioneered by Moran and formalized by Hutchinson, provides explicit formulas for dimensions under specific conditions [2,8]. We first define self-similar sets and present a key result for their dimension calculation.

Definition 2.3 (Box-counting dimensions). Let $E \subset \mathbb{R}^k$ be bounded and let $N(E,\varepsilon)$ denote the minimal number of closed balls of radius ε needed to cover E. The upper and lower box-counting dimensions (also called Minkowski dimensions) are

$$\overline{\dim}_B(E) = \limsup_{\varepsilon\to 0} \frac{\log N(E,\varepsilon)}{\log(1/\varepsilon)}, \underline{\dim}_B(E) = \liminf_{\varepsilon\to 0} \frac{\log N(E,\varepsilon)}{\log(1/\varepsilon)}.$$

If the two coincide we write $\dim_B(E)$ for their common value. Always $\dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E)$.

Definition 2.4 (Packing measure and packing dimension). Let $E \subset \mathbb{R}^k$, $s \geq 0$, $\delta > 0$. Define the packing premeasure

$$\mathcal{P}_\delta^S(E) = sup\{\textstyle\sum_i (diam\ B_i)^s : \{B_i = B(x_i, \rho_i)\} pairwise\ disjoint\ closed\ balls, x_i \in E, \rho_i \leq \delta\},$$

where diam $B_i = 2\rho_i$.

Put $\mathcal{P}_0^s(E) = \lim_{\delta\to 0} \mathcal{P}_\delta^s(E)$ and define the s-dimensional packing measure

$$\mathcal{P}^s(E) = \inf\left\{\sum_j \mathcal{P}_0^s(E_j) : E \subset \bigcup_j E_j\right\}.$$

The *packing dimension* is

$$\dim_P(E) = \inf\{s : P^s(E) = 0\} = \sup\{s : P^s(E) = \infty\}.$$

References: Falconer [2, §3.4, eqs. (3.22)–(3.25)].

Definition 2.5 A set $E \subset \mathbb{R}^k$ is self-similar if it is the unique non-empty compact set satisfying $E = \bigcup_{i=1}^N S_i(E)$, where each $S_i : \mathbb{R}^k \to \mathbb{R}^k$ is a contracting similarity with scaling ratio $r_i \in (0, 1)$.

The dimension of such sets can be determined under a geometric constraint.

The following proposition, named after [2,8], provides a precise formula when the similarities satisfy a separation condition.

Definition 2.6 (Moran–Hutchinson formula under the open set condition (OSC)). Let $E = \bigcup_{i=1}^N S_i(E)$ be a self-similar set generated by contracting similarities with ratios $r_i \in (0, 1)$. If the open set condition (OSC) holds (i.e., there exists an open set $O \subset \mathbb{R}^k$ such that $\bigcup_{i=1}^N \dot{S}_i(O) \subset O$ with disjoint images), the Hausdorff dimension $\dim_H(E) = d$ is the unique solution to the Moran–Hutchinson equation $\sum_{i=1}^N r_i^d = 1$. If all ratios are equal, $r_i = r$, then $d = \frac{\ln N}{\ln(1/r)}$.

Proposition 2.7 (OSC self-similar sets: equality of dimensions)**.** Let $E \subset \mathbb{R}^k$ be a self-similar set generated by similarities with ratios $r_i \in (0, 1)$ satisfying the open set condition (OSC), and let s be the unique solution of $\Sigma_i r_i^s = 1$. Then

$$\dim_H(E) = \dim_B(E) = \overline{\dim}_B(E) = \underline{\dim}_B(E) = s,$$

and moreover $0 < H^s(E) < \infty$.

References: Falconer [2, Ch. 9, Thm. 9.3, eqs. (9.9), (9.11)]; see also Hutchinson [4, §5.2].

This result is critical for analyzing the Smale horseshoe's invariant set, which decomposes locally into self-similar Cantor sets along stable and unstable directions. To construct sets with higher dimensions, we need a mechanism to combine lower-dimensional components. The following theorem, a standard result in fractal geometry, addresses the dimension of Cartesian products.

Theorem 2.8 (Product bounds). Let $A \subset R^m$ and $B \subset R^n$ be Borel (or analytic) sets. Then

$$\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B),$$

$$\dim_H(A \times B) \leq \dim_H(A) + \dim_B(B).$$

References: Falconer [2, Ch. 7, Product formulae 7.2–7.3].

Corollary 2.9. For any set $A \subset R^m$ and any integer $n \geq 0$,

$$\dim_H(A \times [0,1]^n) = \dim_H(A) + n.$$

Proof. By Theorem 2.8,

$$\dim_H(A \times [0,1]^n) \geq \dim_H(A) + \dim_H([0,1]^n) = \dim_H(A) + n.$$

For the upper bound, Theorem 2.8 gives

$\dim_H(A \times [0,1]^n) \leq \dim_H(A) + \dim_B([0,1]^n) = \dim_H(A) + n$, since $\dim_B([0,1]^n) = n$. Hence equality holds.

References: Falconer [2, Ch. 7, Product formulae 7.2–7.3, Cor. 7.4].

Proposition 2.10 (Product additivity for OSC self-similar Cantor sets). Let $C_{r_i} \subset R$ $(i = 1, 2)$ be two-map self-similar Cantor sets with common ratios $r_i \in (0, \frac{1}{2})$ satisfying the open set condition. Then

$$\dim_H(C_{r_1} \cdot C_{r_2}) = \dim_H(C_{r_1}) + \dim_H(C_{r_2}), \dim_H(C_{r_i}) = \frac{\ln 2}{\ln(1/r_i)}.$$

Proof. By the Moran–Hutchinson formula and Proposition 2.7, each C_{r_i} satisfies

$\dim_H(C_{r_i}) = \dim_B(C_{r_i}) = \frac{\ln 2}{\ln(1/r_i)}$. For arbitrary Borel (or analytic) sets A, B one has

the general product bounds (Falconer [2, Ch. 7, Product formulae 7.2–7.3]):

$$\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B), \dim_H(A \times B) \leq \dim_H(A) + \dim_B(B).$$

Applying these with $A = C_{r1}$ and $B = C_{r2}$ and using $\dim B(C_{r2}) = \dim H(C_{r2})$ gives $\dim_H(C_{r1} \times C_{r2}) \leq \dim H(C_{r1}) + \dim_H(C_{r2})$. Together with the lower bound we obtain equality.

References: product bounds — Falconer [2, Ch. 7, Product formulae 7.2–7.3]; equality for OSC self-similar sets — Falconer [2, Ch. 9, Thm. 9.3].

Corollary 2.11 As $r_1, r_2 \in (0, \frac{1}{2})$ vary, the map

$$(r_1, r_2) \longmapsto \dim_H(C_{r_1} \times C_{r_2}) = \frac{\ln 2}{\ln(1/r_1)} + \frac{\ln 2}{\ln(1/r_2)}$$

is continuous with range $(0, 2)$.

2.3. The Smale Horseshoe and the Invariant Set

The Smale horseshoe map, a cornerstone of hyperbolic dynamical systems, provides the primary mechanism for generating fractal sets in our construction. Its invariant set, characterized by a Cantor-like structure, serves as a tunable fractal component whose Hausdorff dimension can be precisely controlled. We first define the horseshoe map and its action on a compact region.

Definition 2.12. The Smale horseshoe map $f : S \to S$ is a diffeomorphism on a compact region $S \subset R^2$ (e.g., the unit square $[0, 1]^2$). It stretches S in one direction (typically vertical) with expansion factor $\lambda > 1$, contracts in another (typically horizontal) with factor $\mu \in (0, 1)$, and folds the resulting set back into S. For a symmetric linear horseshoe with $N = 2$ strips, we set $\mu = 1/\lambda$.

This map's hyperbolic dynamics generate a complex invariant set under iteration. The next definition formalizes this set, which is critical for our dimension-tuning strategy.

Definition 2.13. The invariant set Λ of the Smale horseshoe map f is the set of points that remain in S under all forward and backward iterations:

$$\Lambda = \bigcap_{k\in\mathbb{Z}} f^k(S).$$

For a hyperbolic horseshoe, Λ is a Cantor set with a local product structure, homeomorphic in a neighborhood to the Cartesian product of two Cantor sets C_s and C_u in the stable (contracting) and unstable (expanding) directions, respectively.

Standing assumptions (H) for the model horseshoe. We fix the square $S = [0, 1]^2$ and consider a (piecewise affine) two-strip horseshoe $f_\lambda : S \to S$ with vertical expansion $\lambda > 2$ and horizontal contraction $\mu = 1/\lambda < \frac{1}{2}$, such that:

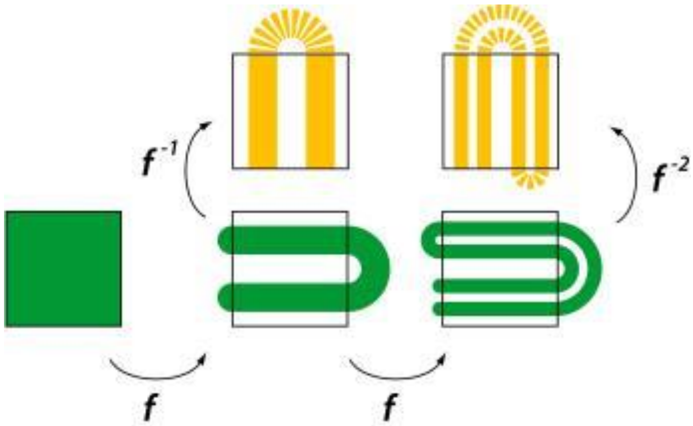

Figure 1: A geometric illustration of the Smale Horseshoe map.

As shown in Figure 1, the map f stretches, contracts, and folds a square, while the inverse map f^{-1} performs the reverse operation, revealing the fractal structure. (Credit: XaosBits, licensed under FAL. Source: Wikimedia Commons)

(H1) $\tilde{f}_\lambda(S) \cap S$ is the disjoint union of two vertical rectangles $V_0, V_1 \subset S$, each of width μ and height 1, whose horizontal projections are $[0,\mu]$ and $[1 — \mu, 1]$; hence the strips are strictly disjoint (equivalently, with $r := \mu = 1/\lambda$, $2r < 1$, i.e. $\lambda > 2$).

(H2) $\tilde{f}_\lambda^{-1}(S) \cap S$ is the disjoint union of two horizontal rectangles $H_0, H_1 \subset S$, each of height μ and width 1, with vertical projections $[0,\mu]$ and $[1—\mu, 1]$ (equivalently, $2r < 1$ with $r = 1/\lambda$, i.e. $\lambda > 2$).

(H3) (Smoothing supported off the invariant set) There exists an open neighborhood $U \subset S$ of the hyperbolic set Λ_λ and a $C^{1+\alpha}$ diffeomorphism f_λ defined on a neighborhood of S such that: (i) $f_\lambda = f_\lambda$ on U (hence, on Λ_λ and a neighborhood of it the dynamics and derivatives coincide with those of the $C^{1+\alpha}$ smoothing that folds the elongated image back into S without creating overlaps across the two Markov branches. In particular, the two-branch Markov structure and the derivative data relevant to the invariant set are unchanged.

(H4) The hyperbolic invariant set $\Lambda_\lambda = \bigcap_{k\in Z} f_\lambda^k(S)$ is a saddle-type horseshoe with local product structure. piecewise-affine model f_λ); (ii) outside U we replace the corners of f_λ by a Remark 2.14 (Parameter regime) . For the one-dimensional slice IFS on (0, 1) the open set condition already holds for $\lambda \geq 2$ (equivalently $r = 1/\lambda \leq$); see Lemma 2.16(i). In the two-dimensional horseshoe, however, we shall work throughout with $\lambda > 2$

so that the two branch rectangles in (H1)–(H2) are strictly separated (i.e., $2r < 1$), avoiding boundary contact at $\lambda = 2$ and yielding a clean two-symbol Markov par- tition. This choice is geometric; the slice-IFS OSC at $\lambda = 2$ is not needed in our construction.

Lemma 2.15 (Smoothing off $\Lambda\lambda$ preserves slice IFS and pressure data)**.** Under (H1), (H2), (H3), we have $f\lambda = f\lambda$ on a neighborhood $U \supset \Lambda\lambda$. Consequently, the one-dimensional first-return maps on local stable/unstable foliations of $\Lambda\lambda$ coincide with those of the piecewise-affine model $f\lambda$, yielding the same two-map similarity IFS with common ratio $r = 1/\lambda$ and the same open set condition. Equivalently, the Holder potentials $\varphi^s = \log // Df\lambda | Es //$ and $\varphi^u = \log // Df\lambda | Eu //$ on $\Lambda\lambda$ agree with those of $f\lambda$, hence the associated pressures are identical. Proof. By (H3), $f\lambda = \tilde{f}\lambda$ on an open neighborhood U of $\Lambda\lambda$. All local (un)stable plaques and their first-return maps are contained in U. Therefore the induced slice dynamics, the two similarity maps with ratio $r = 1/\lambda$, the open set condition, and the thermodynamic potentials restricted to Λ_λ are exactly those of f_λ.

Lemma 2.16 (OSC for slice IFS and the role of the strict gap)**.** Let $r = 1/\lambda$. Consider the two-map IFS on (0, 1) given by $S_0(x) = rx$ and $S_1(x) = rx + (1 - r)$.

(1) **(Slice IFS & OSC)** For $r \leq \frac{1}{2}$ (equivalently $\lambda \geq 2$) we have $S_0(O), S_1(O) \subset O$ and $S_0(O) \cap S_1(O) =$ with $O = (0, 1)$, so the open set condition holds. In particular, at $r = \frac{1}{2}$ the two images are the disjoint open intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.

(2) **(Strict gap for a 2D horseshoe)** To obtain a two-branch horseshoe in $S = [0, 1]^2$ with disjoint Markov rectangles (a strictly positive separation), we impose $2r < 1$, i.e. $\lambda > 2$. This ensures a clean symbolic dynamics and uniform hyperbolicity on the invariant set. Hence, throughout the paper we work under $\lambda > 2$.

Proof. For the slice IFS take O = (0, 1). Then $S0(O) = (0, r)$ and $S1(O) = (1 - r,1)$ are disjoint open intervals whenever $r \leq$ (equivalently $\lambda \geq 2$), so the OSC holds; at r = they meet only at x = in the closure, not in O. For the two-dimensional horseshoe, (H1)–(H2) ensure the two branch rectangles are strictly disjoint precisely. when $2r < 1$ (i.e. $\lambda > 2$), which yields a clean Markov partition and uniform hyperbolicity; hence throughout we assume $\lambda > 2$ for the 2D model, even though the slice-IFS OSC already holds at $\lambda = 2$.

The local product structure of Λ is a key feature. In the general $C^{1+\alpha}$ surface case, Ma~n ́e [6] proved the additivity $\dim_H(\Lambda) = d^s + d^u$, where d^s, d^u are given by pressure equations from thermodynamic formalism. In our linear symmetric two-strip model, each slice is self-similar, so the formula reduces to $\dim_H(\Lambda) = \dim_H(C_s) + \dim_H(C_u)$. This property, combined with the parameter dependence of λ, enables the precise tuning of $\dim_H(\Lambda)$ in our main proof, as detailed in Section 3.

3. Main Result and Constructive Proof

With the foundational tools from geometric measure theory and dynamical systems established in Section 2, we now present the core results of this paper. Our objective is to construct sets with any positive real Hausdorff dimension through a systematic approach. The key insight is to leverage the Smale horseshoe map as a tunable fractal generator, producing an invariant set whose dimension can be precisely controlled within the interval (0, 2). This fractal component is then combined with a Euclidean component via the Cartesian product to achieve the desired dimension. We begin by proving the tunability of the horseshoe's invariant set dimension, followed by the main theorem constructing sets for arbitrary positive dimensions.

3.1. Tunability of the Horseshoe Invariant Set Dimension

We establish that the Smale horseshoe map can generate an invariant set with any Hausdorff dimension in (0, 2) by adjusting its expansion parameter. This result is pivotal for constructing the fractal component required by the main theorem.

Proposition 3.1. Under the standing assumptions (H) for a linear, symmetric two-diffeomorphism f_λ satisfies strip horseshoe with $\lambda > 2$ and $\mu = 1/\lambda < \frac{1}{2}$, the invariant set Λ_λ of the $C^{1+\alpha}$

$$\dim_H(\Lambda_\lambda) = D(\lambda) = \frac{2\ln 2}{\ln\lambda}.$$

Consequently, for any $d_{frac} \in (0, 2)$ there exists $\lambda = 4^{1/dfrac} > 2$ such that $\dim_H(\Lambda\lambda) = d_{frac}$.

Proof. Consider the linear, symmetric two-strip family {fλ(˜)} on S = [0, 1]2 under (H1)–(H2) with λ > 2 a˜nd μ = 1/λ < . By (H3) we take a C1+α diffeomorphism The invariant set Λλ is hyperbolic with local product structure (e.g. [9, Thm. 4.3]), and fλ is C1+α . By Ma˜n´e's additivity on surfaces [6], dimH (Λλ) = ds + du , and by Lemma 2.15 together with Proposition 2.6 the stable/unstable slice dimensions are

$$d_s = d_u = \tfrac{\ln 2}{\ln\lambda}. \text{ Hence } D(\lambda) = \tfrac{2\ln 2}{\ln\lambda}.$$

By Lemma 2.16 and Lemma 2.15, both stable/unstable slices are two-map selfsimilar sets with common ratio r = 1/λ satisfying the OSC, so by Proposition 2.6 we compute ds = $\dim_H$ (Cs) and du = $\dim_H$ (Cu).

(1) Stable Dimension (d_S): The set C_S is formed by N = 2 similarities of common ratio r = 1/λ < $\frac{1}{2}$; hence, by Proposition 2.6

$$d_s = \dim_H(C_s) = \frac{\ln N}{\ln(1/r)} = \frac{\ln 2}{\ln\lambda}.$$

(2) Unstable Dimension (d_U): Similarly,

$$d_u = \dim_H(C_u) = \frac{\ln N}{\ln(1/r)} = \frac{\ln 2}{\ln\lambda}.$$

Summing the dimensions, we obtain the total dimension as a function of λ:

$$D(\lambda) = d_s + d_u = \frac{\ln 2}{\ln\lambda} + \frac{\ln 2}{\ln\lambda} = \frac{2\ln 2}{\ln\lambda}.$$

Continuity. Since lnλ is continuous and strictly positive on (2, ∞), the map λ →} $\frac{1}{\ln\lambda}$ is continuous there. Hence D(λ) = 2ln2 ·(lnλ)$^{-1}$ is continuous on (2, ∞).

Monotonicity. A direct derivative computation shows

$$D'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\frac{2\ln 2}{\ln\lambda}\Big) = -\frac{2\ln 2}{\lambda(\ln\lambda)^2} < 0 \qquad (\lambda > 2),$$

so D is strictly decreasing on (2, ∞).

Range. The endpoint limits are

$$\lim_{\lambda\to 2^+} D(\lambda) = \frac{2\ln 2}{\ln 2} = 2, \lim_{\lambda\to\infty} D(\lambda) = \frac{2\ln 2}{\infty} = 0.$$

Because the domain (2, ∞) is open, the value 2 is not attained (only approached as λ → 2+), and 0 is only a limit as λ → ∞ . Combining the strict monotonicity with these limits yields

$$D\big((2, \infty)\big) = (0, 2).$$

Since we adopt λ > 2 for strict separation (Remark 2.14), the value 2 is not attained and only appears in the limit $\lambda \to 2^+$

Solving for λ (existence and uniqueness). Given any target $d_{frac} \in (0, 2)$, the intermediate value theorem and strict monotonicity imply there exists a unique λ0 ∈ (2, ∞) such that D(λ0) = d_{frac}. Solving

$$d_{\text{frac}} = \frac{2\ln 2}{\ln\lambda_0} \iff \ln\lambda_0 = \frac{2\ln 2}{d_{\text{frac}}} \iff \lambda_0 = \exp\Big(\frac{2\ln 2}{d_{\text{frac}}}\Big) = 4^{\,1/d_{\text{frac}}}.$$

Note that $d_{frac} < 2$ implies $\lambda_0 > 4^{1/2} = 2$, so indeed $\lambda_0 \in (2, \infty)$, and as $d_{frac} \to 0+$ we have $\lambda_0 \to \infty$.

Therefore, for this λ_0 the horseshoe map $f_{\lambda 0}$ yields an invariant set $\Lambda_{\lambda 0}$ with $\dim_H(\Lambda_{\lambda 0}) = d_{frac}$. Moreover, by the branch-preserving smoothing condition (H3) together with Lemma 2.15, the slice IFS and their contraction ratio $r = 1/\lambda$ are unchanged by smoothing near the two branches; hence the above formula for $D(\lambda)$ coincides with that of the piecewise-affine model and is unaffected by the smoothing step. This completes the proof.

3.2. Construction of Any Positive Real Hausdorff Dimension

Having established the tunability of the Smale horseshoe's invariant set dimension in Proposition 3.1, we now construct a set with any positive real Hausdorff dimension. The proof combines the fractal component from the horseshoe with a Euclidean component via the Cartesian product, leveraging the dimension product rule.

Theorem 3.2. For any positive real number $d_{target} > 0$, there exists a set S such that its Hausdorff dimension satisfies $\dim_H(S) = d_{target}$.

Proof. Let $d_{target} > 0$ be the desired Hausdorff dimension. We construct the set S in four explicit steps.

(1) Dimension Decomposition: Decompose dtarget into its integer part n = ld_{target} and fractional part $d_{frac} = d_{target} — n$. By definition, $n \geq 0$ is an integer, and $d_{frac} \in [0, 1)$.

(2) Fractal Component: Construct a set M C R^2 with $\dim_H(M) = d_{frac}$. (2.1). If $d_{frac} = 0$, let M = {p} C R2 be a singleton, so $\dim_H(M) = 0$ by the definition of Hausdorff dimension. (2.2). If dfrac ∈ (0, 1), note that (0, 1) C (0, 2). By Proposition 3.1, there exists a Smale horseshoe map fλ with invariant set M = Λλ C R2 such that $\dim_H(M)$ = dfrac , achieved by setting λ = 41/d_{frac} .

(3) Integer Component: Construct a set E C Rn with $\dim_H(E)$ = n. (3.1). If n = 0, let E = {q} C R0 be a singleton (where R0 denotes a point), so $\dim_H(E) = 0$. In this case, S = M ×E M. (3.2). If n > 0, let E = [0, 1]n C Rn , the n-dimensional unit hypercube, with $\dim_H(E)$ = n (cf. [2]).

(4) Combination and Verification: Define the final set S = M ×E C R2+n. By Corollary 2.9 with E = [0, 1]n , the Hausdorff dimension is:

$$\dim_H(S) = \dim_H(M) + \dim_H(E) = d_{frac} + n = d_{target}.$$

This construction produces a set S with the desired dimension, completing the proof.

The embedding space R^{2+n} ensures the Cartesian product is well-defined, but its dimension does not affect $\dim_H(S)$. Alternative embeddings or choices of M and E may alter topological properties, as discussed in Section 5

4. Constructing Sets with Arbitrary Positive Hausdorff Dimensions via Smale Horseshoe

To illustrate the constructive proof of Theorem 3.2, this section provides explicit constructions of sets with specific Hausdorff dimensions, showcasing the versatility of the four-step method outlined in Section 3. We first present examples for typical non-integer dimensions √2 and π, demonstrating the Smale horseshoe's tunability for moderate fractional parts (Subsection 4.1). We then analyze the expansion parameter λ's behavior and construct a near-boundary example for a dimension close to an integer, highlighting the method's performance under extreme dynamical tuning (Subsection 4.2). These examples illustrate the method's flexibility across a range of target dimensions.

4.1. Construction of Typical Non-integer Dimensions Sets

We present two detailed constructions, applying the method of Theorem 3.2 to target dimensions √2 and π . A summary table compares the key parameters of each construction.

(1) Construction for $d_{target} = \sqrt{2}$:

1) Dimension Decomposition: Compute the integer part $n = \lfloor\sqrt{2}\rfloor = 1$ and fractional part $d_{frac} = \sqrt{2} — 1$.

2) Fractal Component: Since $d_{frac} \in (0, 1) \subset (0, 2)$, apply Proposition 3.1 to construct a Smale

horseshoe map f_λ with invariant set $M = \Lambda_\lambda \subset R^2$ such that $\dim_H(M) = \sqrt{2} — 1$. From Proposition 3.1, set

$$\lambda = 4^{1/\sqrt{2}-1}.$$

3) Integer Component: For n = 1, let $E = [0, 1] \subset R^1$, the unit interval, with $\dim_H(E) = 1$.

4) Combination and Verification: Form $S = M \times E \subset R^3$. By Corollary 2.9,

$$\dim_H(S) = \dim_H(M) + \dim_H(E) = (2 — 1) + 1 = \sqrt{2}.$$

(2) Construction for $d_{target} = \pi$:

1) Dimension Decomposition: Compute the integer part $n = \lfloor\pi\rfloor = 3$ and fractional part $d_{frac} = \pi — 3$.

2) Fractal Component: Since $d_{frac} \in (0, 1) \subset (0, 2)$, apply Proposition 3.1 to construct a Smale horseshoe map f_λ with invariant set $M = \Lambda_\lambda \subset R^2$ such that $\dim_H(M) = \pi — 3$. Set

$$\lambda = 4^{\pi-3}.$$

3) Integer Component: For n = 3, let $E = [0, 1]^3 \subset R^3$, the unit cube, with $\dim_H(E) = 3$.

4) Combination and Verification: Form $S = M \times E \subset R^5$. By Corollary 2.9, $\dim_H(S) = \dim_H(M) + \dim_H(E) = (\pi — 3) + 3 = \pi$.

4.2. Construction of Near-integer Dimension Sets and Λ Parameter Analysis

The examples in Subsection 4.1, constructing sets with non-integer dimensions $\sqrt{2}$ and π, illustrate the Smale horseshoe's ability to tune the fractal component for moderate fractional dimensions, as shown in Table 1. Here, we analyze the behavior of the expansion parameter λ and provide a near-boundary (i.e., near-integer) example for dtarget = 3.01 to demonstrate the method's performance when the fractional part is very small.

Table 1: Summary of constructions for dimensions $\sqrt{2}$ and π.

Parameter	$d_{target} = \sqrt{2}$	$d_{target} = \pi$
Integer Part (n)	1	3
Fractional Part (d_{frac})	$\sqrt{2}$ -1	π-3
Expansion Rate (λ)	$4^{1/\sqrt{2}-1}$	$4^{1/(\pi-3)}$
Fractal Component (M)	$\Lambda_\lambda \subset R^2$	$\Lambda_\lambda \subset R^2$
Integer Component (E)	$[0, 1] \subset R^1$	$[0, 1]^3 \subset R^3$
Final Set ($S = M \times E$)	$\subset R^3$	$\subset R^5$
Hausdorff Dimension ($\dim_H(S)$)	$\sqrt{2}$	π

From Proposition 3.1, the dimension of the Smale horseshoe's invariant set is $D(\lambda) = 2\ln 2/\ln\lambda$. In our construction we only need fractional parts in (0, 1), in which case we set $\lambda = 4^{1/d_{frac}}$ so that $D(\lambda) = d_{frac}$. Over this range, λ is strictly decreasing in d_{frac}, with $\lambda \to \infty$ as $d_{frac} \to 0^+$ and $\lambda \to 4^+$ as $d_{frac} \to 1^-$. Thus, small fractional parts are the genuinely expensive regime (requiring extremely large unstable expansion), while fractional parts close to 1 correspond to moderate values of λ near 4.

To illustrate this behavior, we construct a set with $d_{target} = 3.01$, whose fractional part is $d_{frac} = 0.01$.

Construction for $d_{target} = 3.01$:

(1) Dimension Decomposition: Compute the integer part $n = \lfloor 3.01\rfloor = 3$ and the fractional part $d_{frac} = 3.01 — 3 = 0.01$.

(2) Fractal Component: Since $d_{frac} = 0.01 \in (0, 1)$, apply Proposition 3.1 to construct a Smale

horseshoe map f_λ with invariant set $M = \Lambda_\lambda \subset \mathbb{R}^2$ such that $\dim_H(M) = 0.01$. Set

$$\lambda = 4^{\frac{1}{0.01}} = 4^{100} = 2^{200} \approx 1.6 \times 10^{60}.$$

The set M is a Cantor-like set, totally disconnected with zero Lebesgue measure (cf. [9, §4.3]).

(3) Integer Component: For $n = 3$, let $E = [0, 1]^3 \subset \mathbb{R}^3$, the unit cube, with $\dim_H(E) = 3$.

(4) Combination and Verification: Form $S = M \times E \subset \mathbb{R}^5$. By Corollary 2.9, $\dim_H(S) = \dim_H(M) + \dim_H(E) = 0.01 + 3 = 3.01$.

The resulting set S combines a Cantor-like fractal with a 3-dimensional Euclidean component, embedded in $\mathbb{R}^5$.

This near-integer example requires extreme tuning of λ: here $\lambda = 4^{100} \approx 1.6 \times 10^{60}$, reflecting a very strong unstable expansion when the fractional part d_{frac} is tiny. This highlights the true dynamical cost of approaching an integer dimension in our construction.

5. Examples

The constructive proof in Section 3 and the examples in Section 4 demonstrate a systematic method to achieve any positive real Hausdorff dimension using the Smale horseshoe map as a tunable fractal generator. This section explores alternative approaches to generate the fractal component and discusses the broader implications of the construction. We first examine other systems capable of producing fractal sets with tunable dimensions, highlighting the modularity of our framework. Then, we reflect on the method's generality and potential extensions within dynamical systems and geometric measure theory.

5.1. Alternative Fractal Generators for Fractional Dimension Components

The construction in Theorem 3.2 is modular: the fractional component need not be produced by a horseshoe. Other parameter–dependent fractals can be used, provided we can control their Hausdorff dimension on a target interval. When the desired dimension exceeds the integer dimension of the ambient space, we simply increase the ambient dimension; in practice we take $\mathbb{R}^m$ with $m \geq \lceil d_{target} \rceil$. Concretely, when $d_{frac} \in (0, 1)$ one may work in $\mathbb{R}^{n+1}$ with a 1D fractal factor $\times[0, 1]^n$, while for $d_{frac} \in [1, 2)$ one may work in $\mathbb{R}^{n+2}$ with a 2D fractal factor $\times[0, 1]^n$.

Self-similar Cantor sets (OSC). A versatile replacement of the horseshoe is the generalized two-map Cantor set $C_r \subset \mathbb{R}$: start with $[0, 1]$ and, at each step, remove the open middle interval of length $1 - 2r$, keeping two intervals of length $r \in (0, \frac{1}{2})$. This self-similar set satisfies the open set condition (OSC), and the Moran–Hutchinson formula yields

$$\dim_H(C_r) = \frac{\log 2}{\log(1/r)}, 0 < \mathcal{H}^{\dim_H(C_r)}(C_r) < \infty \text{ [2, 4, 8]}.$$

As $r \uparrow \frac{1}{2}$, $\dim_H(C_r) \uparrow 1$, so $(0, 1)$ is covered continuously. Hence for any $d_{frac} \in (0, 1)$ we may take

$$r = 2^{-1/d_{frac}} \in \left(0, \tfrac{1}{2}\right), \dim_H(C_r) = d_{frac},$$

and use $M = C_r$ as the fractional component in Theorem 3.2.

To cover $(0, 2)$ with purely self-similar factors, consider a product $C_{r_1} \times C_{r_2} \subseteq \mathbb{R}^2$ with $r_i \in (0, \frac{1}{2})$. Since each 1D factor is an OSC self-similar set, one has $\dim_H = \dim_B$ for each factor [2, Thm. 9.3]. Combining this with the general product bounds [2, Eq. (7.6) and Eq. (7.7)] yields the equality

$$\dim_H\left(C_{r_1} \times C_{r_2}\right) = \dim_H(C_{r_1}) + \dim_H(C_{r_2}) = \frac{\log 2}{\log(1/r_1)} + \frac{\log 2}{\log(1/r_2)}.$$

Therefore the range $(0, 2)$ is obtained continuously by varying (r_1, r_2). In particular, for any $d_{frac} \in (1, 2)$ one can choose $\dim_H(C_{r1}) = \dim_H(C_{r2}) = \frac{1}{2}d_{frac}$ (e.g., $r_1 = r_2 = 2^{-2/d_{frac}}$), so that $\dim_H(C_{r1} \times C_{r2}) = d_{frac}$. This 2D Cantor product can replace the horseshoe component M when a 1D Cantor factor does not suffice.

Chaotic attractors as numerical substitutes. Chaotic attractors from other dynamical systems also offer parameter-dependent fractals. For the Hénon map $f(x, y) = (1 - ax^2 + y, bx)$, there are foundational rigorous results on chaotic dynamics and SRB-type behavior [4], but we do not rely on a rigorous Hausdorff-dimension formula; reported "dimension values" in the literature are typically numerical estimates (and in practice may refer to information/correlation/Kaplan-Yorke dimensions). For the classical Lorenz system $(\sigma,\rho,\beta) = (10, 28, 8/3)$, numerical studies also report fractal-dimension estimates near 2 (often around 2.06); see, e.g., Viswanath [9] for an analysis of fractal properties of the Lorenz attractor. These systems thus provide numerically tunable alternatives to the horseshoe, but, unlike the explicit formula in Proposition 3.1, they do not furnish an analytic dimension function covering the full $(0, 2)$ range.

5.2. Generalizations and Future Directions

The alternative generators in Subsection 5.1 illustrate the modularity of our construction, allowing flexibility in the fractal component. Here, we explore broader generalizations, leveraging iterated function systems (IFS) as a general framework for fractal generation, and discuss topological properties and future research directions in dynamical systems and geometric measure theory.

The construction in Theorem 3.2 relies on the additive property of Hausdorff dimensions under Cartesian products (Theorem 2.8). This approach extends beyond the Smale horseshoe and the alternatives in Subsection 5.1. Any parameterdependent IFS producing a fractal set with a continuous dimension function over a sufficient range can replace the horseshoe. For example, an IFS with variable contraction ratios in R^m can generate fractal components with dimensions in $(0, k)$ for some $k > 0$ (cf. [4]). Combining such a component with a Euclidean set in R^n extends the method to dimensions beyond $(0, 2)$, provided the dimension function's continuity is verified.

The topological properties of the constructed sets offer another avenue for generalization. The Smale horseshoe's invariant set Λ_λ is a Cantor set, totally disconnected with zero Lebesgue measure [10]. The final set $S = M \times E$ inherits properties from its components; for instance, modifying E to a fractal set with integer dimension, under conditions ensuring Borel set properties, preserves $\dim_H(S)$ while altering connectedness or compactness [2]. Analyzing the Hausdorff measure of S at its critical dimension could further elucidate its geometric structure.

Future research could investigate the uniqueness of the constructed sets. While Theorem 3.2 ensures existence, comparing sets generated by different systems (e.g., Smale horseshoe versus Hénon map) for the same dimension may reveal variations in symbolic dynamics or stability [10]. Extending the construction to non-Euclidean metric spaces, such as hyperbolic manifolds, could also enhance its applicability in dynamical systems. These directions highlight the synergy between fractal geometry and chaotic dynamics, opening new paths for studying sets with prescribed dimensions.

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